Université Nice-Sophia Antipolis
SMEFA302 - ECUE Probabilistic computational methods, 2024-2025
Sylvain Rubenthaler (`https://math.unice.fr/~rubentha/`)

Mid-term exam (duration : 1h30)

Exercise 1. We have random variables X_1, X_2, ..., X_n ($n \in \mathbb{N}^*$). They all have the same law μ_θ, that depends on a parameter $\theta \in \mathbb{R}$. Suppose T is a function of $X_1, \dots, X_n$. Let $\widehat{\theta}_n$ be an estimator of θ (based on $X_1, \dots, X_n$). We set $\theta_n^* = \mathbb{E}(\widehat{\theta}_n | \sigma(T))$. Show that

$$\mathbb{E}((\theta_n^* - \theta)^2) \leq \mathbb{E}((\widehat{\theta}_n - \theta)^2)\,.$$

Hint: $\mathbb{E}(\theta|\sigma(T)) = \theta$.

$$\begin{array}{rcl}
\mathbb{E}((\theta_n^* - \theta)^2) & = & \mathbb{E}((\mathbb{E}(\widehat{\theta}_n|\sigma(T)) - \theta)^2) \\
 & = & \mathbb{E}((\mathbb{E}(\widehat{\theta}_n - \theta|T))^2) \\
\text{(Jensen)} & \leq & \mathbb{E}(\mathbb{E}((\widehat{\theta}_n - \theta)^2|T)) \\
 & = & \mathbb{E}((\widehat{\theta}_n - \theta)^2)
\end{array}$$

Exercise 2. We are interested in the following SDE (on an interval $[0,T]$, $T > 0$)

$$dX_t = \mu X_t dt + \sigma dW_t\,,\ X_0 = 0\,, \tag{0.1}$$

($(W_t)_{t\geq 0}$ is a standard Brownian motion, μ and σ are > 0). We take $n \in \mathbb{N}^*$. We set $\Delta = T/n$. For all $k \in \{0, 1, \dots, n\}$, $t_k = kT/n$. For all $t \in [0,T]$, $\underline{t} = \sup_{k\in\{0,1,\dots,n\}}\{t_k : t_k \leq t\}$. The Euler scheme associated to Equation (0.1) is defined recursively by

$$Y_0 = 0, Y_{k+1} = Y_k(1 + \mu\Delta) + \sigma\sqrt{\Delta}Z_{k+1}\,,$$

where, for all $k \geq 1$,

$$Z_k = \frac{W_{t_k} - W_{t_{k-1}}}{\sqrt{\Delta}}\,.$$

For all k, Y_k is an approximation of X_{t_k}.

(1) Show that, for all $k \geq 0$,

$$\mathbb{E}(Y_k^2) = \frac{\sigma^2}{\mu}\left(\frac{(1+\mu\Delta)^{2k} - 1}{2 + \mu\Delta}\right)\,.$$

We show this using a recurrence on k.

- True for $k = 0$.
- Suppose this is true in k. The variables Y_k and Z_{k+1} are independant so

$$\begin{array}{rcl}
\mathbb{E}(Y_{k+1}^2) & = & (1+\mu\Delta)^2\mathbb{E}(Y_k^2) + \sigma^2\Delta \\
\text{(recurrence)} & = & (1+\mu\Delta)^2 \times \dfrac{\sigma^2}{\mu}\dfrac{(1+\mu\Delta)^{2k} - 1}{2+\mu\Delta} + \sigma^2\Delta \\
 & = & \dfrac{\sigma^2}{\mu}\dfrac{(1+\mu\Delta)^{2k+2}}{2+\mu\Delta} - \dfrac{\sigma^2}{\mu}\dfrac{(1+\mu\Delta)^2}{2+\mu\Delta} + \sigma^2\Delta \\
 & = & \dfrac{\sigma^2}{\mu}\dfrac{(1+\mu\Delta)^{2k+2}}{2+\mu\Delta} + \dfrac{\sigma^2}{\mu}\left(-\dfrac{(1+\mu\Delta)^2}{2+\mu\Delta} + \mu\Delta\right) \\
 & = & \dfrac{\sigma^2}{\mu}\dfrac{(1+\mu\Delta)^{2k+2}}{2+\mu\Delta} + \dfrac{\sigma^2}{\mu}\left(\dfrac{-(1+\mu\Delta)^2 + \mu\Delta(2+\mu\Delta)}{2+\mu\Delta}\right) \\
 & = & \dfrac{\sigma^2}{\mu} \times \dfrac{(1+\mu\Delta)^{2k+2} - 1}{2+\mu\Delta}\,.
\end{array}$$

(2) Show that, for all $k \geq 0$,

$$X_{t_{k+1}} = e^{\mu\Delta} X_{t_k} + \sigma e^{\mu t_{k+1}} \int_{t_k}^{t_{k+1}} e^{-\mu s} dW_s \,.$$

Hint: introduce $Z_t = e^{\mu t} X_{t_k} + \sigma e^{\mu t} \times \int_{t_k}^{t} e^{-\mu s} dW_s$, for $t \in [t_k, t_{k+1}]$.

We set $k \geq 0$. We set $Z_t = e^{\mu t} X_{t_k} + \sigma e^{\mu t} \times \int_{t_k}^{t} e^{-\mu s} dW_s$, for $t \in [t_k, t_{k+1}]$. Using the product rule

$$\begin{aligned}
Z_t &= Z_0 + \int_{t_k}^{t} \mu e^{\mu s} X_{t_k} ds + \int_{t_k}^{t} \sigma e^{\mu s} e^{-\mu s} dW_t + \int_{t_k}^{t} \left(\int_{t_k}^{s} e^{-\mu u} dW_s \right) \sigma(\mu e^{\mu s}) ds + \langle \sigma e^{\mu \cdot}, \int_{t_k}^{\cdot \cdot} e^{-\mu s} dW_s \rangle_t \\
&= Z_0 + \int_{t_k}^{t} \mu e^{\mu s} X_{t_k} ds + \int_{t_k}^{t} \sigma dW_t + \int_{t_k}^{t} \left(\int_{t_k}^{s} e^{-\mu u} dW_s \right) \sigma(\mu e^{\mu s}) ds + 0 \\
&= Z_0 + \int_{t_k}^{t} \mu Z_s ds + \sigma \int_{t_k}^{t} \sigma dW_t \,.
\end{aligned}$$

The function $x \mapsto x$ is Lipshitz so there is a unique strong solution to the SDE on $[t_k, t_{k+1}]$:

$$U_{t_k} = X_{t_k} \,,\; dU_t = \mu U_t dt + \sigma dW_t \,.$$

So $Z_t = X_t$ for all $t \in [t_k, t_{k+1}\}$. Which imples $Z_{t_{k+1}} = X_{t_{k+1}}$ (the desired result).

(3) Show that, for all $k \geq 0$,

$$\mathbb{E}((Y_{k+1} - X_{t_{k+1}})^2) \leq (1+\Delta) e^{2\mu\Delta} \mathbb{E}((Y_k - X_{t_k})^2) + \left(1 + \frac{1}{\Delta}\right) \mathbb{E}(Y_k^2)(e^{\mu\Delta} - 1 - \mu\Delta)^2 + \sigma^2 \int_0^{\Delta} (e^{\mu u} - 1)^2 du \,.$$

Hint : if A and B are real random variables, then $2\mathbb{E}(AB) \leq \Delta \mathbb{E}(A^2) + \frac{1}{\Delta}\mathbb{E}(B^2)$.

We have

$$(X_{t_{k+1}} - Y_{k+1})^2 = \left(e^{\mu\Delta}(X_{t_k} - Y_k) + Y_k(e^{\mu\Delta} - 1 - \mu\Delta) + \int_{t_k}^{t_{k+1}} \sigma(e^{\mu(t_{k_1} - s)} - 1) dW_s \right)^2 \,.$$

We have

$$\mathbb{E}\left(\int_{t_k}^{t_{k+1}} \sigma(e^{\mu(t_{k_1} - s)} - 1) dW_s \right) = 0$$

(because we integrate a deterministic process against dW_s). The term $\int_{t_k}^{t_{k+1}} \sigma(e^{\mu(t_{k_1} - s)} - 1) dW_s$ is independant of Y_k, X_{t_k}. So

$$\begin{aligned}
\mathbb{E}((X_{t_{k+1}} - Y_{k+1})^2) &= e^{2\mu\Delta} \mathbb{E}((X_{t_k} - Y_k)^2) + (e^{\mu\Delta} - 1 - \mu\Delta)^2 \mathbb{E}(Y_k^2) \\
&\quad + 2e^{\mu\Delta}(e^{\mu\Delta} - 1 - \mu\Delta) \mathbb{E}((X_{t_k} - Y_k) Y_k) + \mathbb{E}\left(\left(\int_{t_k}^{t_{k+1}} \sigma(e^{\mu(t_{k_1} - s)} - 1) dW_s \right)^2 \right) \\
&= e^{2\mu\Delta} \mathbb{E}((X_{t_k} - Y_k)^2) + (e^{\mu\Delta} - 1 - \mu\Delta)^2 \mathbb{E}(Y_k^2) \\
&\quad + 2e^{\mu\Delta}(e^{\mu\Delta} - 1 - \mu\Delta) \mathbb{E}((X_{t_k} - Y_k) Y_k) + \mathbb{E}\left(\int_{t_k}^{t_{k+1}} \sigma^2 (e^{\mu(t_{k_1} - s)} - 1)^2 ds \right) \\
&\leq e^{2\mu\Delta} \mathbb{E}((X_{t_k} - Y_k)^2) + (e^{\mu\Delta} - 1 - \mu\Delta)^2 \mathbb{E}(Y_k^2) + (\Delta \mathbb{E}(e^{2\mu\Delta}(X_{t_k} - Y_k)^2) + \frac{1}{\Delta}(e^{\mu\Delta} - 1 - \mu\Delta)^2 \mathbb{E}(Y_k^2)) \\
&\quad + \sigma^2 \int_0^{\Delta} (e^{\mu u} - 1)^2 du \\
&= (1+\Delta) e^{2\mu\Delta} \mathbb{E}((Y_k - X_{t_k})^2) + \left(1 + \frac{1}{\Delta}\right) \mathbb{E}(Y_k^2)(e^{\mu\Delta} - 1 - \mu\Delta)^2 + \sigma^2 \int_0^{\Delta} (e^{\mu u} - 1)^2 du.
\end{aligned}$$